1) $\vec{F} = \langle -xy^2, x^2y, e^{5z} \rangle$. (Note: this problem takes place in $\mathbb{R}^3$)

a) Show that $\nabla \times \vec{F}$ is tangent to the cylinder

$$x^2 + y^2 = 1.$$

b) Compute

$$\oint_C \vec{F} \cdot d\vec{r}$$

where C is $\vec{r}(t) = \langle \cos t, \sin t, \sin^3(t)\cos^4(t) \rangle$

$$0 \le t \le 2\pi.$$

Hint: a) To show tangency, show dot prod. w/ normal to surface is zero.

2) Let S be the portion of the elliptic paraboloid

$$z = x^2 + 4y^2 - 4$$

that is underneath the xy-plane, with the downwards orientation. Let

$$\vec{F} = \langle y \log_2(x^2+4y^2+z^2) + 3x^2y^2\cos(x^3),$$
$$-3x + 2y\sin(x^3),$$
$$e^{yz}\arctan(y^{x^2+1})\rangle.$$

Compute $\displaystyle\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$.

Hints: pretty clear indicator that Stokes is involved.

ex) Surface $z = x^2 + y^2$ inside $x^2 + y^2 \leq 4$

Suppose we want to compute

$$\iint_D 1 \, dS$$

Method 1:

$\vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$

$\vec{r}_x = \langle 1, 0, 2x \rangle$

$\vec{r}_y = \langle 0, 1, 2y \rangle$

$\vec{r}_x \times \vec{r}_y = \langle -2x, -2y, 1 \rangle$

$$\iint_D 1 \, dS = \iint_{x^2+y^2 \leq 4} \sqrt{4x^2+4y^2+1} \, dx\,dy.$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2+1} \; r \, dr \, d\theta$$

this part is just Ch 15 stuff.

Method 2:

$\vec{r}(x,y) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$

(apologies for multiple uses of r...)

$\vec{r}_r = \langle \cos\theta, \sin\theta, 2r \rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$\vec{r}_r \times \vec{r}_\theta = \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle$

$$\iint_D 1 \, dS = \int_0^{2\pi} \int_0^2 \sqrt{4r^4(\cos^2\theta + \sin^2\theta) + r^2} \; dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2+1} \; r \, dr \, d\theta$$

⚠ Note that in second method, we did not introduce "another" factor of r (as in the first method $dxdy = \underline{\underline{r}}\, dr\, d\theta$) b/c it was already accounted for.

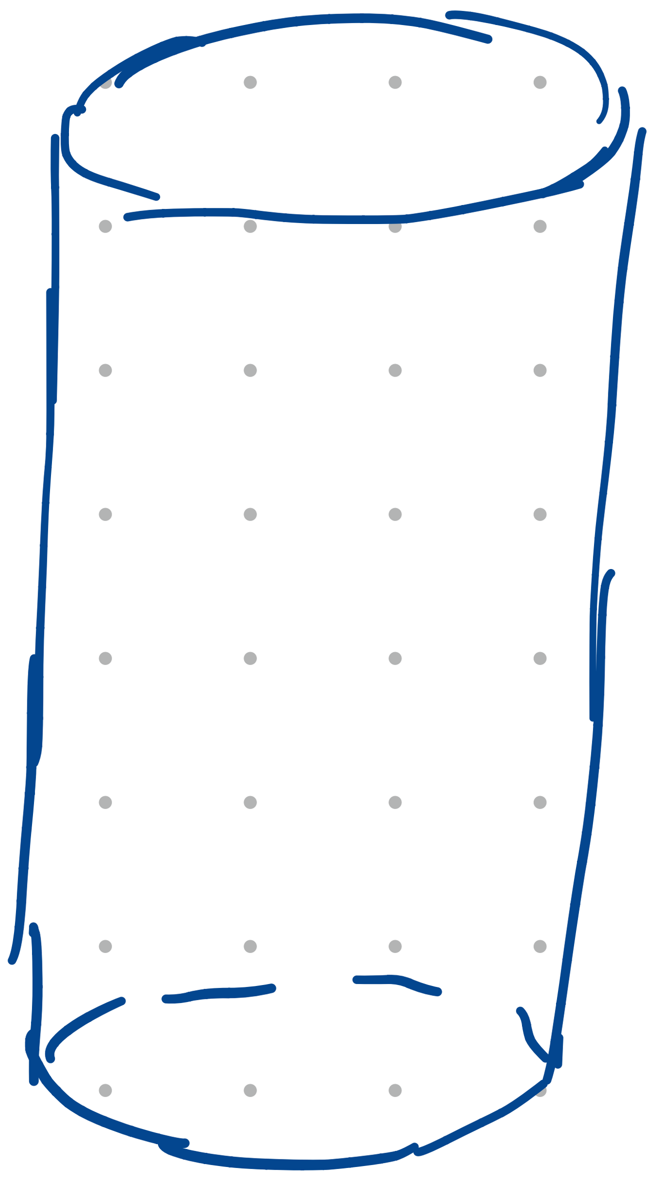
Solution to 1)

a) $\nabla \times \vec{F} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -xy^2 & x^2y & e^{5z} \end{bmatrix}$

$= \langle 0 - 0,\ 0 - 0,\ 2xy - (-2xy) \rangle$

$= \langle 0, 0, 4xy \rangle$

(at this point, you are probably geometrically convinced of (a), but let's check algebraically)



a normal vec to cylinder:

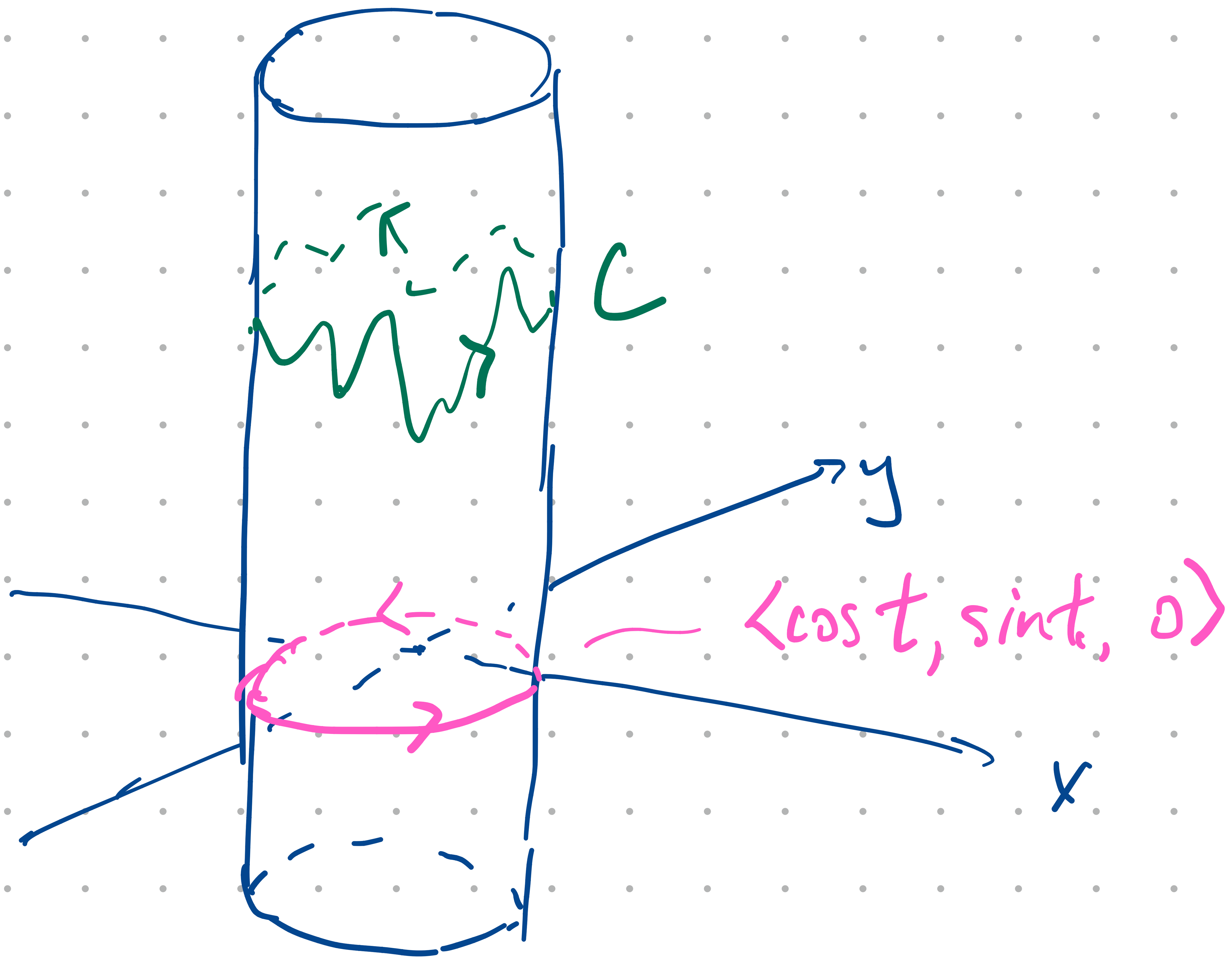
$\langle 2x, 2y, 0 \rangle = \nabla(x^2+y^2-1)$

$\langle 0,0,4xy \rangle \cdot \underline{\langle 2x, 2y, 0 \rangle} = 0.$

note that this is not a unit normal, but that's not relevant to the problem.

b) Observe from (a) that $\nabla \times \vec{F}$ has zero flux through any region of the cylinder.

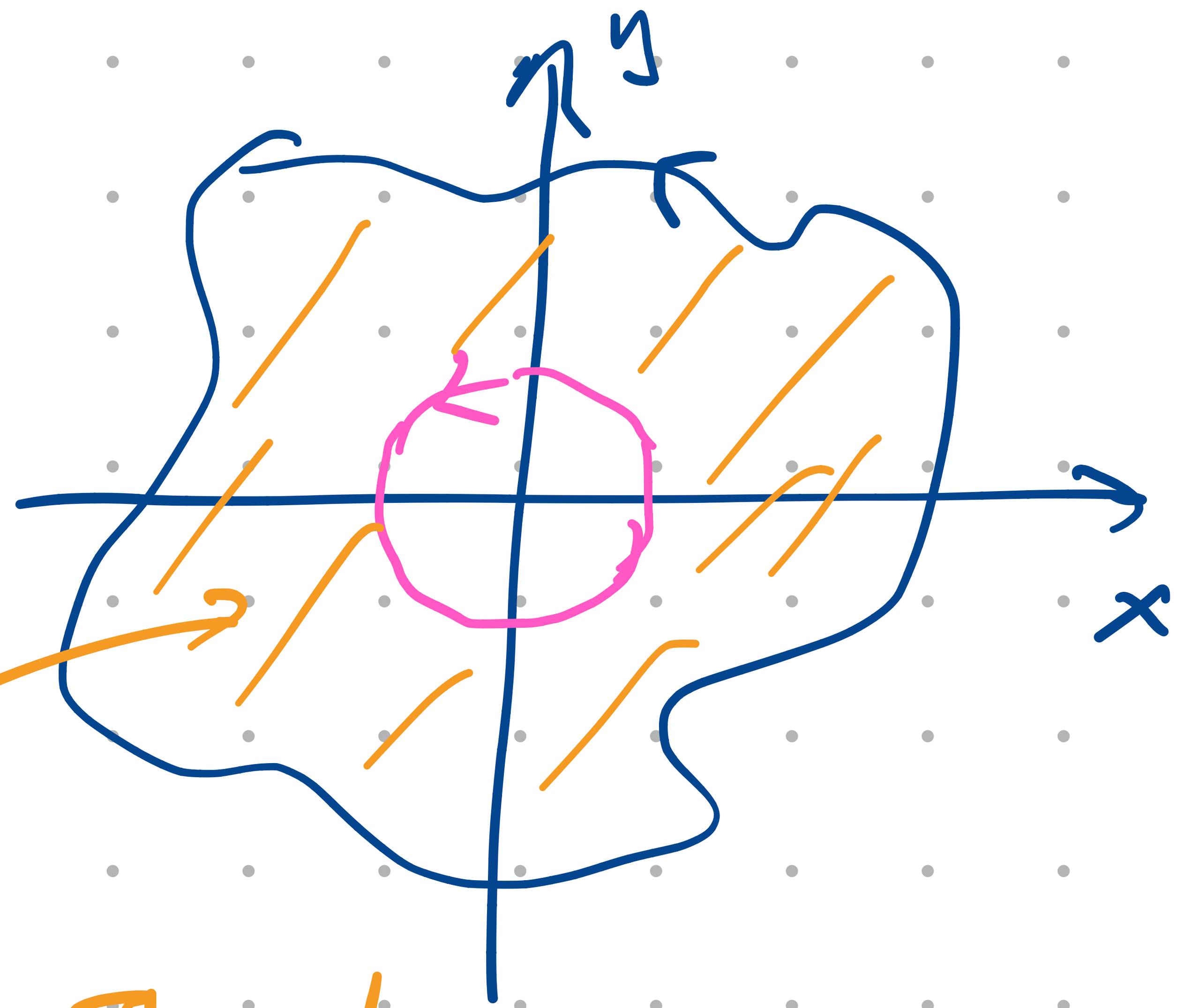
Also: C lies on the cylinder.



Claim:

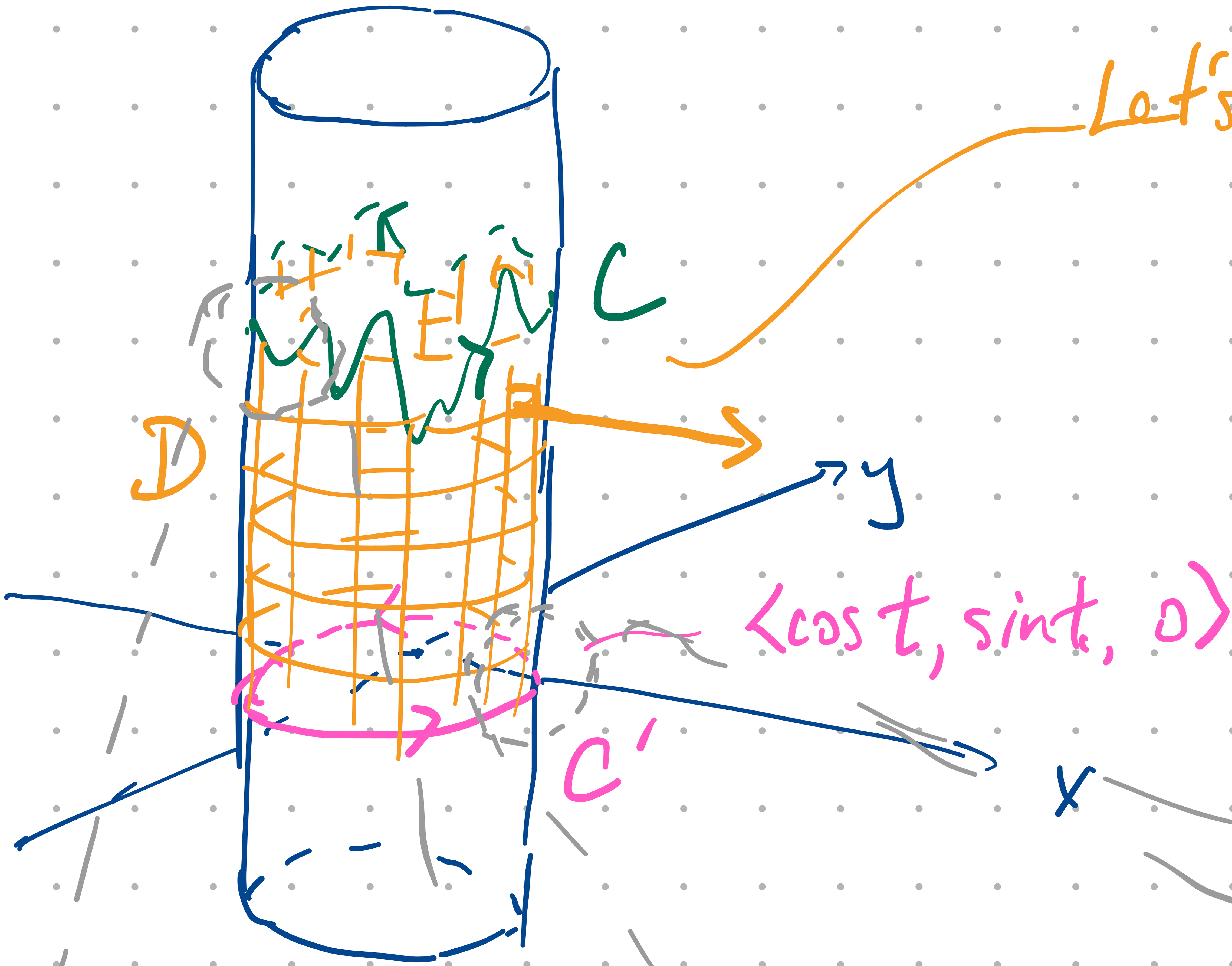
$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{\circlearrowleft} \vec{F} \cdot d\vec{r}.$$

Rmk. This problem is similar in spirit to p. 1150 #38 :



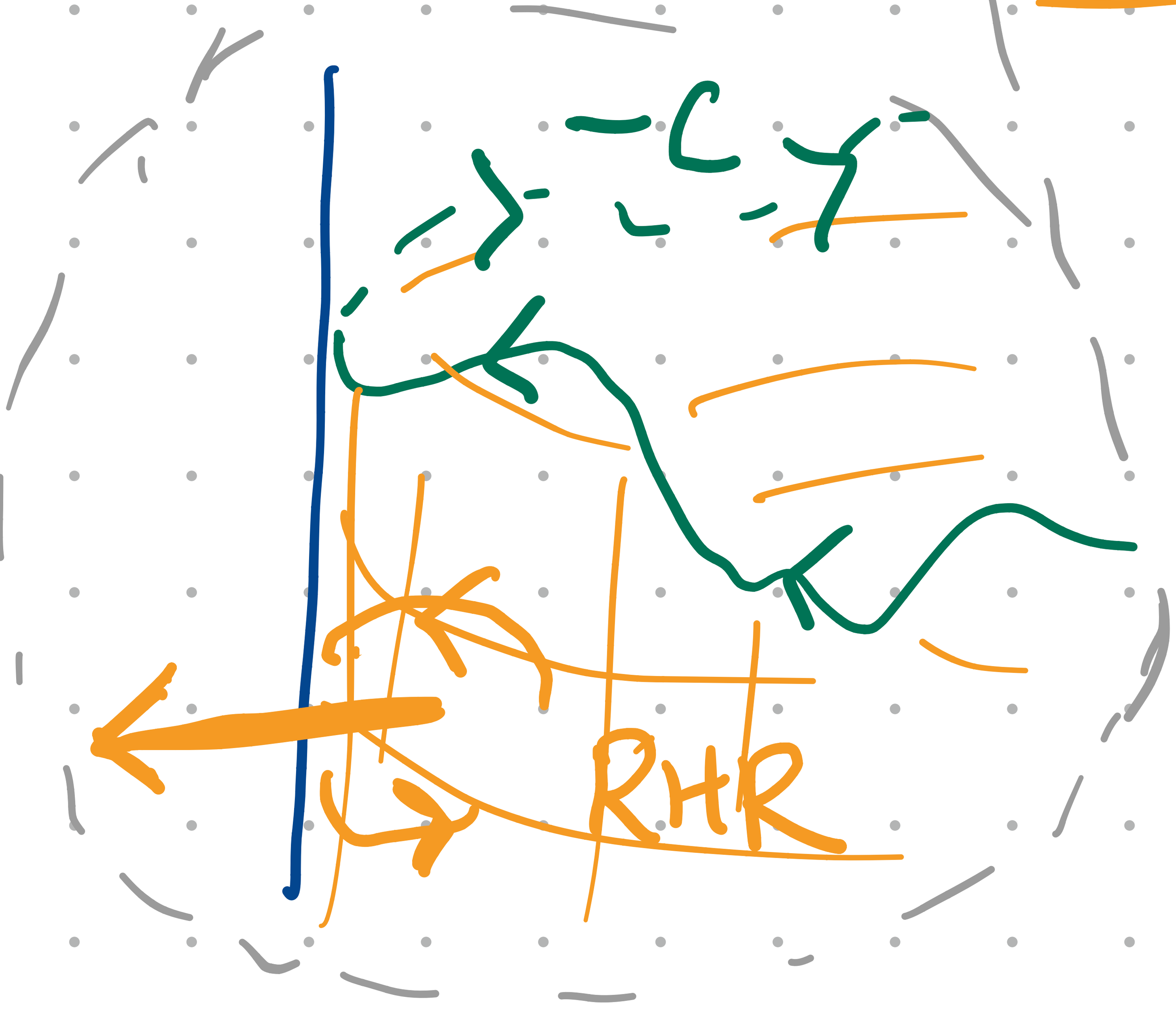
We applied Green's Thm here.

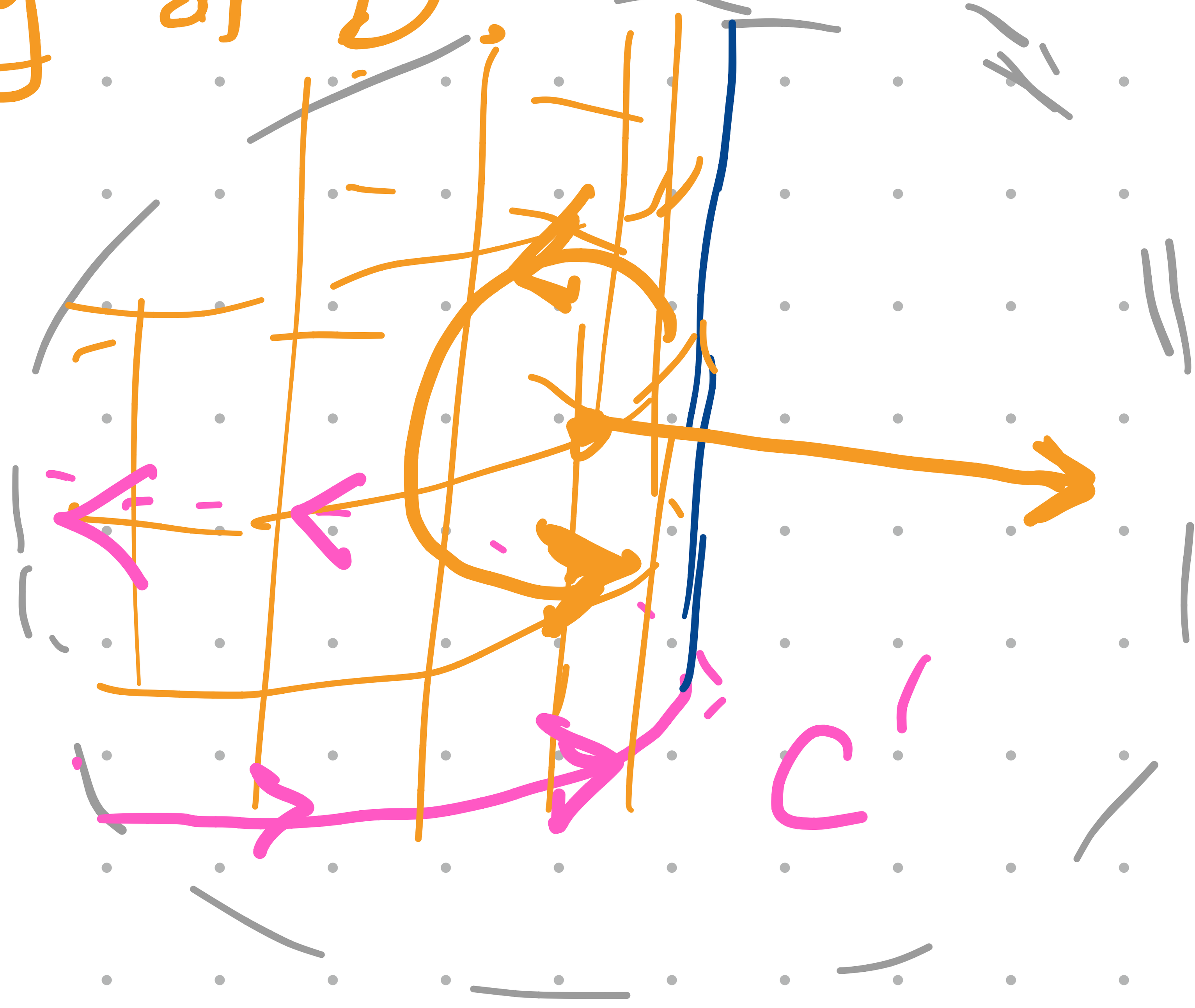
Similarly:



Let's see what happens if I apply Stokes here.

What is the oriented boundary of D?





So Stokes says:

$$\iint_D (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{-C} \vec{F} \cdot d\vec{r} + \int_{C'} \vec{F} \cdot d\vec{r}$$

$$\|$$

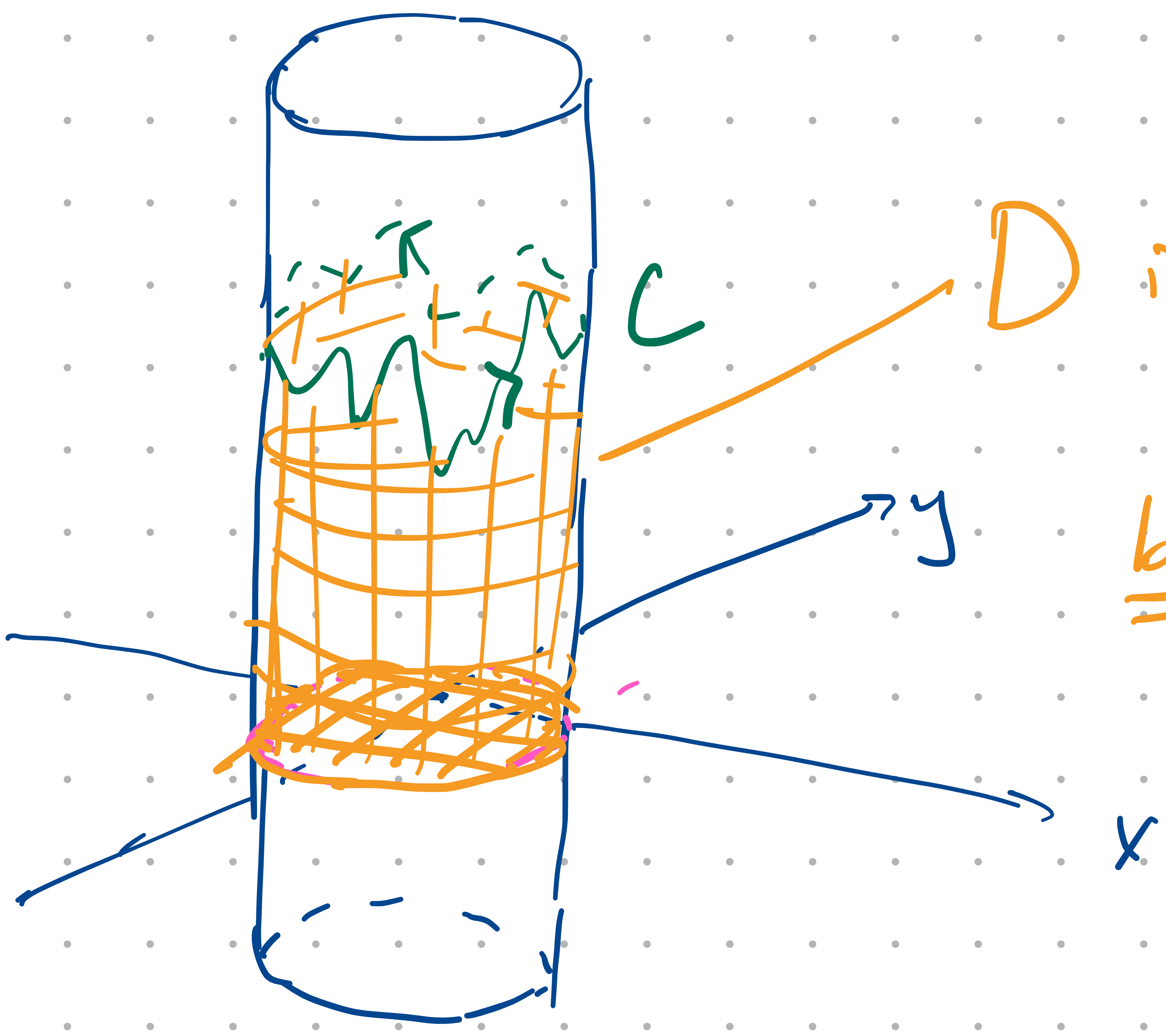
0 from (2), where we saw
$(\nabla \times \vec{F}) \cdot d\vec{S} = 0$.

thus $$\int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r}$$ as claimed.

two ways to proceed from here: either directly plug in parametrization for C';

or switch to double integral of $\nabla \times \vec{F}$ through bottom face of cylinder.

~~Alternative~~ method:
Better



D is cylinder + bottom **but no top**

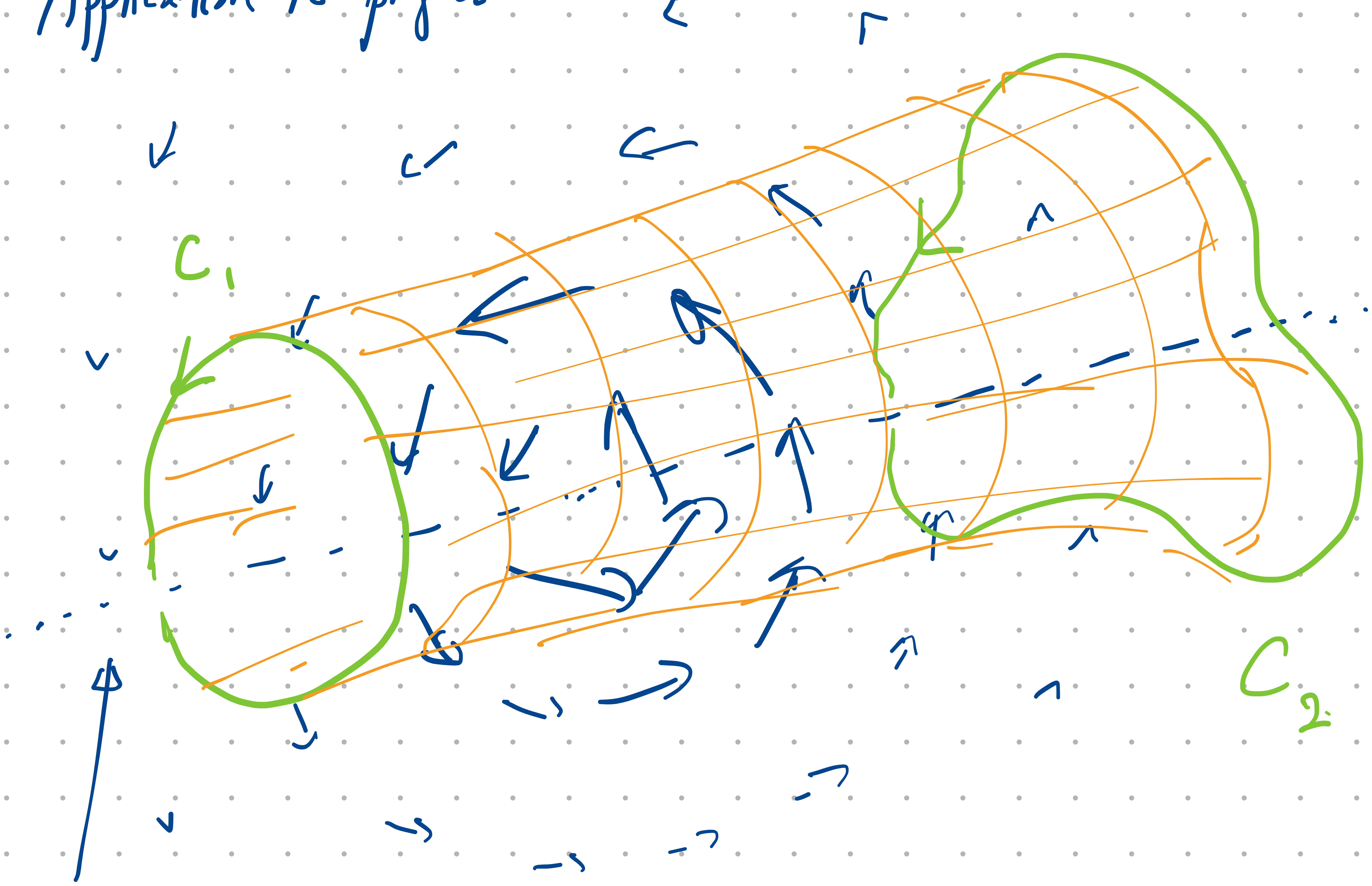
Apply Stokes here.

Conclusion:

$$\int_C \vec{F}\cdot d\vec{r} = \iint_{\text{bottom}} (\nabla\times\vec{F})\cdot d\vec{S}.$$

$$+ \cancelto{0}{\iint_{\text{side}} (\nabla\times\vec{F})\cdot d\vec{S}} \quad \text{by part (a)}$$

Application to physics:



imagine this is a wire carrying some current.

not defined on the wire

It generates a magnetic field $\vec{B}$.

Under some mild assumptions: $\nabla \times \vec{B} = 0$

Claim: $\int_{C_1} \vec{B} \cdot d\vec{r} = \int_{C_2} \vec{B} \cdot d\vec{r}$ (pf: use Stokes on )

This result manifests in Ampère's law in physics.